therefore, by the first process in the sequence, which is the transfer of energy from the largest eddies. These eddies have energy of order u_0^2 and timescale $\tau_0 = \ell_0/u_0$, and so the rate of energy transfer can be supposed to scale as $u_0^2/\tau_0 = u_0^3/\ell_0$. Consequently, consistent with the experimental observations in free shear flows, this picture of the cascade indicates that ε scales as u_0^3/ℓ_0, independent of ν (at the high Reynolds numbers being considered).

6.1.2 Kolmogorov Hypotheses

Several fundamental questions remain unanswered. What is the size of the smallest eddies that are responsible for dissipating the energy? As ℓ decreases, do the characteristic velocity and timescales $u(\ell)$ and $\tau(\ell)$ increase, decrease or remain the same? (The assumed decrease of the Reynolds number $u(\ell)\ell/\nu$ with ℓ is not sufficient to determine these trends.)

These questions and more are answered by the theory advanced by Kolmogorov (1941b) [1] which is stated in the form of three hypotheses. A consequence of the theory—which Kolmogorov used to motivate the hypotheses—is that both the velocity and timescales $u(\ell)$ and $\tau(\ell)$ decrease as ℓ decreases.

The first hypothesis concerns the isotropy of the small-scale motions. In general, the large eddies are anisotropic and are affected by the boundary conditions of the flow. Kolmogorov argued that in the chaotic scale-reduction process, by which energy is transferred to successively smaller and smaller eddies, the directional biases of the large scales are lost. Hence (approximately stated):

> **Kolmogorov's Hypothesis of Local Isotropy.** At sufficiently high Reynolds number, the small-scale turbulent motions ($\ell \ll \ell_0$) are statistically isotropic.

(The term "local isotropy" means isotropy only at small scales, and is defined more precisely in Section 6.1.4.) It is useful to introduce a lengthscale ℓ_{EI} (with $\ell_{EI} \approx \frac{1}{6}\ell_0$, say) as the demarcation between the anisotropic large eddies ($\ell > \ell_{EI}$) and the isotropic small eddies ($\ell < \ell_{EI}$). (Justification for this specification of ℓ_{EI}, and of other scales introduced below, is provided in Section 6.5.)

Just as the directional information of the large scales is lost as the energy passes down the cascade, Kolmogorov argued that all information about the

[1] An English translation of this paper is reproduced as Kolmogorov (1991) in a special issue of the Proceedings of the Royal Society published to mark the 50th anniversary of the original publication. The other papers in this issue, which relate to the Kolmogorov hypotheses, are also of interest.

geometry of the large eddies—as determined by the mean flow field and boundary conditions—is also lost. As a consequence, the statistics of the small-scale motions are in a sense universal—similar in every high-Reynolds number turbulent flow.

On what parameters does this statistically-universal state depend? In the energy cascade (for $\ell < \ell_{EI}$) the two dominant processes are the transfer of energy to successively smaller scales, and viscous dissipation. A plausible hypothesis, then, is that the important parameters are the rate at which the small scales receive energy from the large scales (which we denote by $\mathcal{T}_{EI}$), and the kinematic viscosity ν. As we shall see, the dissipation rate ε is determined by the energy transfer rate $\mathcal{T}_{EI}$, so that these two rates are nearly equal, i.e., $\varepsilon \approx \mathcal{T}_{EI}$. Consequently, the hypothesis that the statistically-universal state of the small scales is determined by ν and the rate of energy transfer from the large scales $\mathcal{T}_{EI}$ can be stated as:

> **Kolmogorov's First Similarity Hypothesis**. In every turbulent flow at sufficiently high Reynolds number, the statistics of the small-scale motions ($\ell < \ell_{EI}$) have a universal form that is uniquely determined by ν and ε.

The size range $\ell < \ell_{EI}$ is referred to as the *universal equilibrium range*. In this range, the timescales $\ell/u(\ell)$ are small compared to ℓ_0/u_0, so that the small eddies can adapt quickly to maintain a dynamic equilibrium with the energy transfer rate $\mathcal{T}_{EI}$ imposed by the large eddies.

Given the two parameters ε and ν, there are (to within multiplicative constants) unique length, velocity and time scales that can be formed. These are the Kolmogorov scales:

$$\eta \equiv (\nu^3/\varepsilon)^{\frac{1}{4}}, \tag{6.1}$$

$$u_\eta \equiv (\varepsilon\nu)^{\frac{1}{4}}, \tag{6.2}$$

$$\tau_\eta \equiv (\nu/\varepsilon)^{\frac{1}{2}}. \tag{6.3}$$

Two identities stemming from these definitions clearly indicate that the Kolmogorov scales characterize the very smallest, dissipative eddies. First, the Reynolds number based on the Kolmogorov scales is unity, i.e., $\eta u_\eta/\nu = 1$, which is consistent with the notion that the cascade proceeds to smaller and smaller scales until the Reynolds number $u(\ell)\ell/\nu$ is small enough for dissipation to be effective. Second, the dissipation rate is given by

$$\varepsilon = \nu(u_\eta/\eta)^2 = \nu/\tau_\eta^2, \tag{6.4}$$

showing that $(u_\eta/\eta) = 1/\tau_\eta$ provides a consistent characterization of the velocity gradients of the dissipative eddies.

Having identified the Kolmogorov scales, we can now state a consequence of the hypotheses that demonstrates their potency, and clarifies the meaning of the phrases "similarity hypothesis" and "universal form". Consider a point $\mathbf{x}_0$ in a high-Reynolds-number turbulent flow at a time t_0. In terms of the Kolmogorov scales at $(\mathbf{x}_0, t_0)$, non-dimensional coordinates are defined by

$$\mathbf{y} = (\mathbf{x} - \mathbf{x}_0)/\eta, \tag{6.5}$$

and the non-dimensional velocity-difference field is defined by

$$\mathbf{w}(\mathbf{y}) = [\mathbf{U}(\mathbf{x}, t_0) - \mathbf{U}(\mathbf{x}_0, t_0)]\,/u_\eta. \tag{6.6}$$

It is not possible to form a non-dimensional parameter from ε and ν; and so (on dimensional grounds) the "universal form" of the statistics of the non-dimensional field $\mathbf{w}(\mathbf{y})$ cannot depend on ε and ν. Consequently, according to the Kolmogorov hypotheses stated above, when examined on not too large a scale (specifically $|\mathbf{y}| < \ell_{EI}/\eta$), the non-dimensional velocity field $\mathbf{w}(\mathbf{y})$ is statistically isotropic and statistically identical at all points $(\mathbf{x}_0, t_0)$ in all high-Reynolds-number turbulent flows. *On the small scales, all high-Reynolds number turbulent velocity fields are statistically* similar; *that is, they are statistically identical when scaled by the Kolmogorov scales (Eqs. 6.5 and 6.6).*

The ratios of the smallest to largest scales are readily determined from the definitions of the Kolmogorov scales and from the scaling $\varepsilon \sim u_0^3/\ell_0$. The results are:

$$\eta/\ell_0 \sim \mathrm{Re}^{-\frac{3}{4}}, \tag{6.7}$$

$$u_\eta/u_0 \sim \mathrm{Re}^{-\frac{1}{4}}, \tag{6.8}$$

and

$$\tau_\eta/\tau_0 \sim \mathrm{Re}^{-\frac{1}{2}}. \tag{6.9}$$

Evidently, at high Reynolds number, the velocity and time scales of the smallest eddies (u_η and τ_η) are—as previously supposed—small compared to those of the largest eddies (u_0 and τ_0).

Inevitably, and as is evident from flow visualization (e.g., Fig. 1.2 on page 5), the ratio η/ℓ_0 decreases with increasing Re. As a consequence, at sufficiently high Reynolds-number, there is a range of scales ℓ that are very small compared to ℓ_0, and yet very large compared to η, i.e., $\ell_0 \gg \ell \gg \eta$. Since eddies in this range are much bigger than the dissipative eddies, it may be supposed that their Reynolds number $\ell u(\ell)/\nu$ is large, and consequently that their motion is little affected by viscosity. Hence, following from this and from the first similarity hypothesis, we have (approximately stated):

> **Kolmogorov's Second Similarity Hypothesis.** In every turbulent flow at sufficiently high Reynolds number, the statistics of the motions of scale ℓ in the range $\ell_0 \gg \ell \gg \eta$ have a universal form that is uniquely determined by ε, independent of ν.

It is convenient to introduce a lengthscale ℓ_{DI} (with $\ell_{DI} = 60\eta$, say), so that the range in the above hypothesis can be written $\ell_{EI} > \ell > \ell_{DI}$. This lengthscale ℓ_{DI} splits the universal equilibrium range ($\ell < \ell_{EI}$) into two subranges: the *inertial subrange* ($\ell_{EI} > \ell > \ell_{DI}$); and the *dissipation range* ($\ell < \ell_{DI}$). As the names imply, according to the second similarity hypothesis, motions in the inertial subrange are determined by inertial effects—viscous effects being negligible—whereas only motions in the dissipation range experience significant viscous effects, and so are responsible for essentially all of the dissipation. The different lengthscales and ranges are sketched in Fig. 6.1. (We shall see that the bulk of the energy is contained in the larger eddies in the size range $\ell_{EI} = \frac{1}{6}\ell_0 < \ell < 6\ell_0$, which is therefore called the *energy-containing range*. The suffixes "EI" and "DI" indicate that ℓ_{EI} is the demarcation line between energy (E) and inertial (I) ranges, as ℓ_{DI} is that between the dissipation (D) and inertial (I) subranges.)

Length, velocity and time scales cannot be formed from ε alone. But given an eddy size ℓ (in the inertial subrange), characteristic velocity and time scales for the eddy are those formed from ε and ℓ:

$$u(\ell) = (\varepsilon\ell)^{\frac{1}{3}} = u_\eta(\ell/\eta)^{\frac{1}{3}} \sim u_0(\ell/\ell_0)^{\frac{1}{3}}, \tag{6.10}$$

$$\tau(\ell) = (\ell^2/\varepsilon)^{\frac{1}{3}} = \tau_\eta(\ell/\eta)^{\frac{2}{3}} \sim \tau_0(\ell/\ell_0)^{\frac{2}{3}}. \tag{6.11}$$

A consequence, then, of the second similarity hypothesis is that (in the inertial subrange) the velocity and timescales $u(\ell)$ and $\tau(\ell)$ decrease as ℓ decreases.

In the conception of the energy cascade, a quantity of central importance—denoted by $\mathcal{T}(\ell)$—is the rate at which energy is transferred from eddies larger

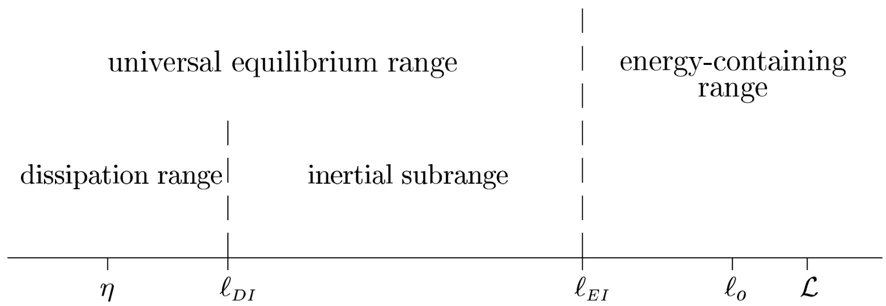

Figure 6.1: Eddy sizes ℓ (on a logarithmic scale) at very high Reynolds number showing the different lengthscales and ranges.

than ℓ to those smaller than ℓ. If this transfer process is accomplished primarily by eddies of size comparable to ℓ, then $\mathcal{T}(\ell)$ can be expected to be of order $u(\ell)^2/\tau(\ell)$. Stemming from Eqs. (6.10) and (6.11), the identity

$$u(\ell)^2/\tau(\ell) = \varepsilon, \tag{6.12}$$

is particularly revealing, therefore, since it suggests that $\mathcal{T}(\ell)$ is independent of ℓ (for ℓ in the inertial subrange). As we shall see, this is the case, and furthermore $\mathcal{T}(\ell)$ is equal to ε. Hence we have

$$\mathcal{T}_{EI} \equiv \mathcal{T}(\ell_{EI}) = \mathcal{T}(\ell) = \mathcal{T}_{DI} \equiv \mathcal{T}(\ell_{DI}) = \varepsilon, \tag{6.13}$$

(for $\ell_{EI} > \ell > \ell_{DI}$). That is, the rate of energy transfer from the large scales, $\mathcal{T}_{EI}$, determines: the constant rate of energy transfer through the inertial subrange, $\mathcal{T}(\ell)$; hence the rate at which energy leaves the inertial subrange and enters the dissipation range $\mathcal{T}_{DI}$; and hence the dissipation rate ε. This picture is sketched in Fig. 6.2.

6.1.3 Energy Spectrum

It remains to be determined how the turbulent kinetic energy is distributed among the eddies of different sizes. This is most easily done for homogeneous turbulence by considering the energy spectrum function $E(\kappa)$ introduced in Chapter 3 (Eq. 3.166).

Recall from Section 3.7 that motions of lengthscale ℓ correspond to wavenumber $\kappa = 2\pi/\ell$, and that the energy in the wavenumber range (κ_a, κ_b)

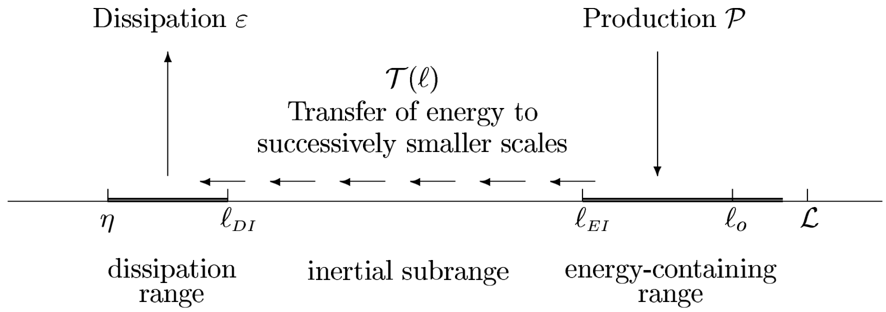

Figure 6.2: Schema of the energy cascade at very high Reynolds number.

is

$$k_{(\kappa_a,\kappa_b)} = \int_{\kappa_a}^{\kappa_b} E(\kappa)\,\mathrm{d}\kappa. \tag{6.14}$$

In Section 6.5, $E(\kappa)$ is considered in some detail, and one result of interest here is that the contribution to the dissipation rate ε from motions in the range (κ_a, κ_b) is

$$\varepsilon_{(\kappa_a,\kappa_b)} = \int_{\kappa_a}^{\kappa_b} 2\nu\kappa^2 E(\kappa)\,\mathrm{d}\kappa. \tag{6.15}$$

It follows from Kolmogorov's first similarity hypothesis that in the universal equilibrium range ($\kappa > \kappa_{EI} \equiv 2\pi/\ell_{EI}$) the spectrum is a universal function of ε and ν. And from the second hypothesis it follows that in the inertial range ($\kappa_{EI} < \kappa < \kappa_{DI} \equiv 2\pi/\ell_{DI}$) the spectrum is

$$E(\kappa) = C\varepsilon^{\frac{2}{3}}\kappa^{-\frac{5}{3}}, \tag{6.16}$$

where C is a universal constant. (These assertions are justified in Section 6.5.)

To understand some basic features of the Kolmogorov $-\frac{5}{3}$ spectrum, we consider the general power-law spectrum

$$E(\kappa) = A\kappa^{-p}, \tag{6.17}$$

where A and p are constants. The energy contained in wavenumbers greater than κ is

$$k_{(\kappa,\infty)} \equiv \int_{\kappa}^{\infty} E(\kappa')\,\mathrm{d}\kappa' = \frac{A}{p-1}\kappa^{-(p-1)}, \tag{6.18}$$

for $p > 1$, while the integral diverges for $p \leq 1$. Similarly the dissipation in wavenumbers less than κ is

$$\varepsilon_{(0,\kappa)} \equiv \int_0^{\kappa} 2\nu\kappa'^2 E(\kappa')\,\mathrm{d}\kappa' = \frac{2\nu A}{3-p}\kappa^{3-p}, \tag{6.19}$$

for $p < 3$, while the integral diverges for $p \geq 3$. Thus, $p = \frac{5}{3}$, corresponding to the Kolmogorov spectrum, is around the middle of the range $(1, 3)$ for which the integrals $k_{(\kappa,\infty)}$ and $\varepsilon_{(0,\kappa)}$ converge. The amount of energy in the high wavenumbers decreases as $k_{(\kappa,\infty)} \sim \kappa^{-\frac{2}{3}}$ as κ increases, while the dissipation in the low wavenumbers decreases as $\varepsilon_{(0,\kappa)} \sim \kappa^{\frac{4}{3}}$ as κ decreases towards zero.

While the Kolmogorov $-\frac{5}{3}$ spectrum applies only to the inertial range, the observations made are consistent with the notion that the bulk of the energy is in the large scales ($\ell > \ell_{EI}$ or $\kappa < 2\pi/\ell_{EI}$), and that the bulk of the dissipation is in the small scales ($\ell < \ell_{DI}$ or $\kappa > 2\pi/\ell_{DI}$).

6.1.4 Restatement of the Kolmogorov Hypotheses

In order to deduce precise consequences from them, it is worthwhile to provide here more precise statements of the Kolmogorov (1941) hypotheses. Kolmogorov presented these in terms of an N-point distribution in the four-dimensional $\mathbf{x}$-t space. Here, however, we consider the N-point distribution in physical space ($\mathbf{x}$) at a fixed time t—which is sufficiently general for most purposes.

Consider a simple domain $\mathcal{G}$ within the turbulent flow, and let $\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(N)}$ be a specified set of points within $\mathcal{G}$. New coordinates and velocity differences are defined by

$$\mathbf{y} = \mathbf{x} - \mathbf{x}^{(0)}, \tag{6.20}$$

and

$$\mathbf{v}(\mathbf{y}) = \mathbf{U}(\mathbf{x}, t) - \mathbf{U}(\mathbf{x}^{(0)}, t), \tag{6.21}$$

and the joint PDF of $\mathbf{v}$ at the N points $\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \ldots, \mathbf{y}^{(N)}$ is denoted by f_N.

Definition of Local Homogeneity. The turbulence is locally homogeneous in the domain $\mathcal{G}$, if for every fixed N and $\mathbf{y}^{(n)}$ ($n = 1, 2, \ldots, N$), the N-point PDF f_N is independent of $\mathbf{x}^{(0)}$ and $\mathbf{U}(\mathbf{x}^{(0)}, t)$.

Definition of Local Isotropy. The turbulence is locally isotropic in the domain $\mathcal{G}$ if it is locally homogeneous and if in addition the PDF f_N is invariant with respect to rotations and reflections of the coordinate axes.

Hypothesis of Local Isotropy. In any turbulent flow with a sufficiently large Reynolds number ($\mathrm{Re} = \mathcal{UL}/\nu$), the turbulence is, to a good approximation, locally isotropic if the domain $\mathcal{G}$ is sufficiently small (i.e., $|y^{(n)}| \ll \mathcal{L}$, for all n) and is not near the boundary of the flow or its other singularities.

First Similarity Hypothesis. For locally isotropic turbulence, the N-point PDF f_N is uniquely determined by the viscosity ν and the dissipation rate ε.

Second Similarity Hypothesis. If the moduli of the vectors $\mathbf{y}^{(m)}$ and of their differences $\mathbf{y}^{(m)} - \mathbf{y}^{(n)}$ ($m \neq n$) are large compared to the Kolmogorov scale η, then the N-point PDF f_N is uniquely determined by ε and does not depend on ν.

It is important to observe that the hypotheses apply specifically to velocity differences. The use of the N-point PDF f_N allows the hypotheses to be applied to any turbulent flow, whereas statements in terms of wavenumber spectra apply only to flows that are statistically homogeneous (in at least one direction).

For inhomogeneous flows, local isotropy is possible only "to a good approximation" (as stated in the hypothesis). For example, taking $\mathbf{y}^{(1)} = \mathbf{e}\ell$ and $\mathbf{y}^{(2)} = -\mathbf{e}\ell$ (where ℓ is a specified length and $\mathbf{e}$ a specified unit vector), we have

$$\begin{aligned} \langle \mathbf{v}(\mathbf{y}^{(1)}) - \mathbf{v}(\mathbf{y}^{(2)}) \rangle &= \langle \mathbf{U}(\mathbf{y}^{(1)}) \rangle - \langle \mathbf{U}(\mathbf{y}^{(2)}) \rangle \\ &\approx 2\frac{\ell}{\mathcal{L}} \mathbf{e} \cdot \mathcal{L} \nabla \langle \mathbf{U} \rangle. \end{aligned} \tag{6.22}$$

Evidently this simple statistic is not exactly isotropic, but instead has a small anisotropic component—of order $\ell/\mathcal{L}$—arising from large-scale inhomogeneities.

6.2 Structure Functions

To illustrate the correct application of the Kolmogorov hypotheses we consider—as did Kolmogorov (1941b)—the second-order velocity structure functions.